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On Plasma-Magnetic Shocks

by

Marian H. Rose

March 15, 1960

AEC Research and Development Report

NEW YORK UNIVERSITY

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Magneto-Fluid Dynamics Division
Institute of Mathematical Sciences
New York University

PHYSICS AND
MATHEMATICS

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Abstract

The following report is concerned with the possible existence of a shock in a high-temperature, collisionless plasma situated in a magnetic field. The underlying equations are those of conservation of mass, momentum and energy for each component coupled with Maxwell's equations. The stress tensor is assumed to be anisotropic, its exact form being derived from guiding-center theory. The conclusion is that no shock solution is contained in this system of equations.

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ON PLASMA-MAGNETIC SHOCKS

1. Introduction.

A fully ionized fluid, subjected to a magnetic field, can be expected to sustain a steady-state shock wave if the usual damping mechanisms such as viscosity and heat conduction are present. This is the case, for example, at low temperatures where the mean free path for collisions is small compared to the dimensions of interest and hence the concepts of viscosity and heat conduction are meaningful. This is no longer the case, however, at temperatures sufficiently high so that the mean free path becomes larger than all significant dimensions. Under these conditions, the possible existence of a shock is not at all obvious and, in fact has not yet been proved.

In this report, we shall examine the properties of a set of equations appropriate for a high temperature plasma in a magnetic field. The usual Navier-Stokes stress terms which, in an ordinary fluid, are responsible for dissipation will be replaced by terms derived from guiding center theory. We shall show that these equations do not possess a shock solution and that the only possible transition is a pulse.

Collision-dominated or "magnetohydrodynamic" shocks as they are usually called are the natural starting point for investigating shock phenomena in ionized fluids since they more closely resemble the well-known hydrodynamical shocks. The earliest studies were concerned with deriving the jump conditions and led to the discovery of three possible shock speeds.[1] Although a fully ionized fluid is composed of positively and negatively charged particles (ions and electrons) in equal numbers, these investigations were all based on a one-fluid approximation; this led to manageable equations whose mathematical structure closely resembled that of ordinary fluid dynamics. Moreover, a one-fluid theory is a reasonable approximation at sufficiently low temperatures and high densities to enable the ions and electrons to reach thermodynamic equilibrium through collisions.

The problem of finding a smooth transition curve joining the constant state ahead to the different constant state behind the shock was first considered by Marshall.[2] This again was a one-fluid theory. In addition to the usual Navier-Stokes terms, a finite conductivity term was included; obviously, this led to a shock thickness dependent on mean free path.

Since the concepts of heat conductivity and viscosity lose their meaning at the high temperatures encountered in thermonuclear research, a one-fluid theory was developed which only included electrical resistivity as a dissipative mechanism[3] (electrical resistivity is still meaningful for very large mean free path). Although this is a sub-case of Marshall's it is possibly a better, albeit still a poor approximation to high temperature ionized fluids.

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[1] F. de Hoffmann and E. Teller, Physical Review, 80, "Magneto-Hydrodynamic Shocks", 692 (1950).
K. O. Friedrichs and H. Kranzer, NYO-6486, "Non-linear Wave Motion in Magneto-Hydrodynamics", (1958).

[2] W. Marshall, "The Structure of Magneto-Hydrodynamic Shock Waves", Proceedings of the Royal Society, A, 233, 357 (1956).

Only a two-fluid theory can adequately account for the phenomena occurring in a collisionless ionized fluid. There, if a shock is to occur, the dissipative mechanism must somehow be provided by the cooperative effects arising between charged particles. A sufficient condition for the presence of these effects is that the Debye length be small compared to any reference length of interest. Such a medium was named a "plasma" by Langmuir; the collisionless shocks that might occur within it will be called here "plasma-magnetic" in distinction to the collision-dominated "magneto-hydrodynamic" shocks.

The simplest two-fluid theories, developed by Adlam and Allen[4] and Davis, Lüst and Schlüter[5] were based on the assumption of a zero temperature plasma; they showed that such media were unable to sustain a shock wave, the only possible solutions being solitary pulses. The next simplest theories for non-zero temperature plasmas due to Gardner, Grad, et al., and to Banos and Vernon[6] ontain essentially the same results. More specifically, by starting from the conservation equations with isotropic pressure terms and Maxwell's equations, they have shown that the direction field of this system contains a closed singularity; thus, the only possibilities are a solitary wave or an undamped, periodic oscillation. An interesting result is that the wave length of these oscillations, namely, $L = \sqrt{\frac{m_e}{\mu n}}\ \frac{1}{e}$ (m_e = electron mass, μ = permeability, e = electrical charge, n = number density) is a possible reference length for shock thickness.

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[3] Rose, M. H., "On the Structure of a Steady Hydromagnetic Shock", NYO-7693, (1956).

[4] Adlam, J. H. and Allen J. E., "The Structure of Collision-Free Hydromagnetic Waves", Phil. Mag. 3, 448-456, (1958).

[5] Davis, L., Lüst, R., Schlüter, A., "The Structure of Hydromagnetic Shock Waves: I. Non-Linear Hydromagnetic Waves in a Cold Plasma", Zeitschrift für Naturforschung, 13a, 916-36 (1958).

By introducing an heuristic, collision-dependent friction term proportional to the relative velocity of the two components, Morawetz[7] has obtained a smooth transition from the state ahead of the shock to a damped oscillatory state behind. As expected, the shock width is proportional to L. However, since the real purpose was to investigate the possible existence of shocks in infinite mean free path plasmas, Morawetz[8] subsequently developed a completely collision-free theory. By expanding the particle trajectories and the self-consistent fields in terms of the electron-ion mass ratio, Morawetz has demonstrated to lowest order the existence of a smooth transition from the state ahead of the "shock" to an undamped, oscillatory state behind.

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[6] C. S. Gardner, H. Grad, et al., "Hydromagnetic Shock Waves in High Temperature Plasmas" (Papers presented at the Second International Conference on the Peaceful Uses of Atomic Energy, Geneva, September 1958), NYO-2538.

Baños, A., Jr., and Vernon, R., "Large Amplitude Waves in a Collision-Free Plasma: I. Single Pulses with Isotropic Pressure". (Physics/Plasma Dynamics Research/LRL - No. 1, Preprint of paper submitted to Il Nuovo Cimento).

[7] Morawetz, C. S., "Magnetohydrodynamic Shock Structure Using Friction", NYO-8677 (1959).

A different approach to the problem[*], described in the following report, uses conservation equations of mass, momentum and energy in which collision-independent stress terms replace the usual Navier-Stokes terms as the source of a possible dissipative mechanism. This system of equations is, therefore, an extension of the system with isotropic stress tensor described earlier by Gardner, Grad, et al.[6] The solutions of both systems, however, turn out to be essentially the same. In both cases the singularity behind the shock is closed which points to a pulse-like solution. This is not surprising in view of the fact that the equations are symmetric with respect to the space variable. The shape of the solution profile is of interest since a pulse that has a structure might be interpreted as an incipient shock. More precisely, it is conceivable that the larger the number

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8. Morawetz, C. S., "Magneto-hydrodynamic Shock Structure without Collisions", NYO-2885 (March 1960).

[*] A preliminary study of this problem was described by the author in TID 7503 (1955).

of moments taken into account, the more structure will appear in the pulse with the result that the point where it returns to any initial value will gradually recede from the corresponding initial point.

By using an expansion method suggested by C. S. Gardner it is shown that, to second order in shock strength, the pulse has no structure.

Acknowledgments.

The author takes great pleasure in thanking Prof. Harold Grad for numerous discussions of this problem. She wishes expecially to thank Dr. Clifford Gardner, whose insight and constructive interest were invaluable.

2. Basic Assumptions and Equations.

We shall consider a fully ionized fluid consisting of positive ions and electrons in sufficient numbers to insure charge neutrality to a high degree of approximation. On the other hand the temperature of the fluid should be sufficiently high for the collision mean free path to be infinite. Since the Debye Length is $h = (KT/n_e e^2)^{1/2}$ (K = Boltzmann constant, n_e = electron density, T = temperature in degrees Kelvin), n_e and T must also be of the right magnitudes to make h small compared to any dimension of interest; otherwise, cooperative effects cannot occur.

We assume that the motion of the fluid is confined to the xy plane, that the magnetic field B only has a z component and that x is the only independent variable (see Figure 1). This geometry allows the choice of a coordinate

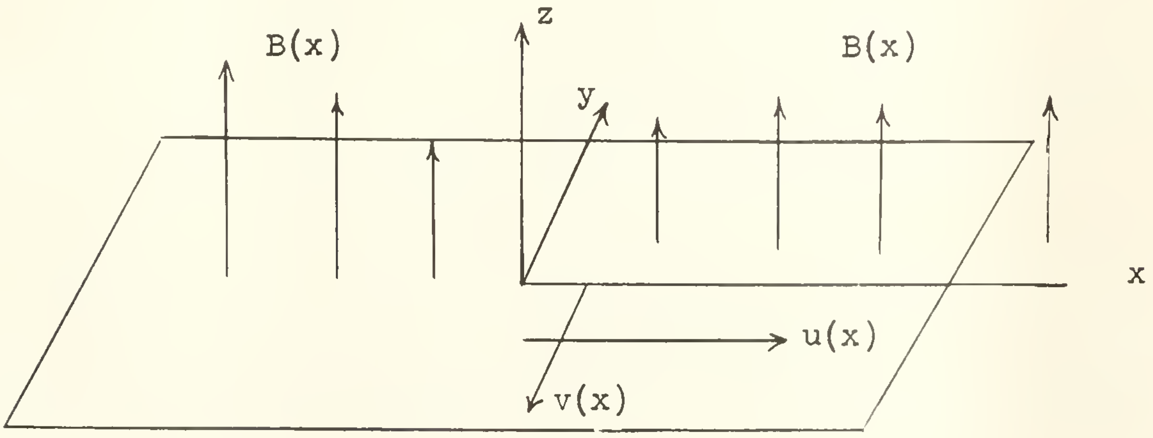

Fig. 1

frame moving at the shock speed and such that the constant velocities of the plasma at either end of the shock are in the x direction only. More specifically, the reference system is chosen so that $-\infty$ is "ahead" of the shock and $+\infty$ is "behind".

The motion of the fluid is governed by the equations of conservation of mass, momentum and energy together with Maxwell's equations.[9] These are (using rationalized M.K.S. units),

Conservation of mass:

$$\frac{d}{dx}(\rho_r u_r) = 0 \tag{2.1}$$

Conservation of momentum:

$$\frac{d}{dx}(\rho_r u_r^2 + P_r^{xx}) = q_r(E_x + v_r B) \tag{2.2}$$

for the x direction

$$\frac{d}{dx}(\rho_r u_r v_r + P_r^{yx}) = q_r(E_y - u_r B) \tag{2.3}$$

for the y direction

Conservation of energy:

$$\frac{d}{dx}(\bar{e}_r u_r + u_r P_r^{xx} + v_r P_r^{yx} + Q_r^x) = q_r(u_r E_x + v_r E_y) \tag{2.4}$$

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[9] Grad, H., Notes on Magneto-Hydrodynamics, IV. NYO-6486, New York University (1956).

Maxwell's equations:

(2.5) $$\operatorname{curl} \frac{B}{\mu} = J$$

(2.6) $$\operatorname{curl} E = 0$$

where

(2.7) $$J = \sum_r q_r v_r \quad .$$

The subscripts "r" may refer either to the ions "i" or to the electrons "e".

The notation has the following meaning:

ρ_r = mass density = $m_r n_r$

m_r = mass of a particle of the r^{th} component.

n_r = number density of the r^{th} component.

q_r = charge density (signed) = en_r.

u_r = x component of macroscopic flow velocity.

v_r = y component of macroscopic flow velocity.

P_r^{ij} = stress tensor = $p_r \delta^{ij} + p_r^{ij}$.

p_r = pressure.

$\bar{e}_r$ = total energy/unit volume = $e_r + \frac{1}{2} \rho_r (u_r^2 + v_r^2)$.

e_r = internal energy/unit volume.

Q_r = heat flow.

E_x = x component of electric field.

E_y = y component of electric field.

B = magnetic field.

J = current.

μ = permeability.

This system of equations is not yet complete since it contains more variables than equations. The extra relations that are needed are

1) The perfect gas law $p_r = \rho_r R T_r$

2) Expressions for the stress and the heat flow in terms of the fluid variables occurring in the equations of motion. These, therefore, will replace the usual Navier-Stokes terms. They are

$$p_r^{xy} = \frac{1}{2\omega_r} p_r \frac{du_r}{dx} = p_r^{yx} \tag{2.8}$$

$$p_r^{xx} = p_r - \frac{p_r}{2\omega_r} \frac{dv_r}{dx} \tag{2.9}$$

$$p_r^{yy} = p_r + \frac{p_r}{2\omega_r} \frac{dv_r}{dx} \tag{2.10}$$

where $\omega_r = eB/m_r$.

Equations (2.9) and (2.10) lead to $p_r = \frac{1}{2}(p_r^{xx} + p_r^{yy})$, i.e., the fluid is strictly 2-dimensional and therefore has a 2-dimensional stress tensor; p_r^{zz} is irrelevant.

The heat flux vector $\vec{Q}_r$ is

$$\vec{Q}_r = (0,\ Q_r^y,\ 0) \text{ where } Q_r^y = \frac{5}{2} \frac{p_r}{\omega_r} \frac{dRT_r}{dx} . \tag{2.11}$$

Since Q_r^y does not occur in the equations of motion, the heat flow term will be entirely absent. The derivation of equations (2.8)-(2.11) is described in Appendix I.

3) The internal energy relation for a 2-dimensional gas, namely

$$e_r = \rho_r R T_r .$$

With these relations, our system of equations will consist of 10 equations with 10 unknowns.

Two simplifications are immediate; first, curl E = 0 yields E_y = constant $\equiv$ F. Secondly, the assumption of charge neutrality ($n_i = n_e$) and the equation $(\text{curl } \frac{B}{\mu})x = J = 0$ lead to $u_e = u_i \equiv u$. Further simplifications are possible if, instead of using separate conservation equations for each component, we use the total equations together with those for either one of the components (e.g., the ions). Since total momentum and energy are conserved, it is obvious that these equations may be integrated once.

The system now has the following form:

Mass

(2.12) Total $\rho u = M$

(2.13) Ions $\rho_i u = M_i$

Momentum

Ions

$$\frac{d}{dx}\left(M_i u + p_i - \frac{p_i}{2\omega_i}\frac{dv_i}{dx}\right) = q_i(E + v_i B) \tag{2.14}$$

$$\frac{d}{dx}\left(M_i v_i + \frac{p_i}{2\omega_i}\frac{du}{dx}\right) = q_i(F - uB) \tag{2.15}$$

Total

$$Mu + p + \frac{B^2}{2\mu} - \frac{p_i}{2\omega_i}\frac{dv_i}{dx} + \frac{p_e}{2\omega_e}\frac{dv_e}{dx} = \Pi_x \tag{2.16}$$

$$M_i v_i + M_e v_e + \frac{1}{2}\left(\frac{p_i}{\omega_i} - \frac{p_e}{\omega_e}\right)\frac{du}{dx} = \Pi_y \tag{2.17}$$

Energy

Ions

$$\frac{d}{dx}\left[2p_i u + \frac{1}{2} M_i(u^2+v_i^2) + \frac{p_i v_i}{2\omega_i}\frac{du}{dx} - \frac{p_i u}{2\omega_i}\frac{dv_i}{dx}\right] = q_i(uE + v_i F) \tag{2.18}$$

Total

$$2\rho u + \frac{1}{2} Mu^2 + \frac{1}{2} M_e v_e^2 + \frac{1}{2} M_e v_i^2 + \frac{FB}{\mu} - \frac{u}{2}\left(\frac{p_i}{\omega_i}\frac{dv_i}{dx} - \frac{p_e}{\omega_e}\frac{dv_e}{dx}\right) + \frac{1}{2}\left(\frac{v_i p_i}{\omega_i} - \frac{v_e p_e}{\omega_e}\right)\frac{du}{dx} = \mathcal{E} \tag{2.19}$$

Maxwell

$$\frac{1}{\mu}\frac{dB}{dx} = q_i(v_e - v_i). \tag{2.20}$$

In order to simplify the notation, E has been used instead of E_x and B instead of B_z. This notation will be adhered to throughout the remainder of the report. Also $\rho = \rho_i + \rho_e$,

$p = p_i + p_e$ and $M = M_i + M_e$ where M_i and M as well as $\mathcal{E}$, Π_x, F and Π_y are all constants of integration. Fortunately, these six constants can be reduced to two by introducing dimensionless variables and by using a coordinate system in which $\Pi_y = 0$. The dimensionless variables are

$$\bar{u} = \frac{M_i u}{\Pi_x}, \qquad \bar{p}_i = \frac{p_i}{\Pi_x}, \qquad \bar{p}_e = \frac{p_e}{\Pi_x}, \qquad \bar{B}^2 = \frac{B^2}{\mu \Pi_x}, \qquad \bar{v}_i = \frac{M_i}{\Pi_x} v_i,$$

$$\bar{v}_e = \frac{M_i}{\Pi_x} v_e, \qquad \bar{\rho}_i = \frac{\Pi_x}{M_i^2} \rho_i, \qquad \bar{\rho}_e = \frac{\Pi_x}{M_i^2} \rho_e, \qquad \bar{F} = \frac{M_i}{\sqrt{\mu}\, \Pi_x^{3/2}} F,$$

$$\bar{E} = \frac{M_i}{\sqrt{\mu}\, \Pi_x^{3/2}} E, \qquad \bar{\mathcal{E}} = \frac{M_i}{\Pi_x^2} \mathcal{E}, \qquad \bar{x} = \frac{\omega_i M_i}{\Pi_x} x, \qquad \bar{q}_i = \frac{\sqrt{\mu}\, \Pi_x^{3/2}}{\omega_i M_i^2} q_i,$$

$$\bar{M}_e = \frac{M_e}{M_i}, \qquad \bar{\omega}_e = \frac{\omega_e}{\omega_i} = \frac{1}{M_e}$$

where, since $M_e \ll M_i$, we have used $\frac{M}{M_i} = 1$. From here on only dimensionless quantities will be used and, therefore, the bars may be omitted without any possibility of confusion.

In its final, dimensionless form our system becomes:

Momentum

$$\frac{du}{dx} + \frac{dp_i}{dx} - \frac{p_i}{2} \frac{d^2 v_i}{dx^2} - \frac{1}{2} \frac{dp_i}{dx} \frac{dv_i}{dx} = q_i (E + v_i B) \tag{2.21}$$

$$\frac{dv_i}{dx} + \frac{p_i}{2} \frac{d^2 u}{dx^2} + \frac{1}{2} \frac{du}{dx} \frac{dp_i}{dx} = q_i (F - uB) \tag{2.22}$$

(2.23) $$u + p + \frac{B^2}{2} - \frac{p_i}{2}\frac{dv_i}{dx} + \frac{p_e}{2\omega_e}\frac{dv_e}{dx} = 1$$

(2.24) $$v_i + M_e v_e + \frac{1}{2}\left(p_i - \frac{p_e}{\omega_e}\right)\frac{du}{dx} = 0$$

Energy

(2.25) $$2p_i \frac{du}{dx} + 2u \frac{dp_i}{dx} + u \frac{du}{dx} + v_i \frac{dv_i}{dx} - \frac{p_i u}{2}\frac{d^2 v_i}{dx^2} + \frac{p_i v_i}{2}\frac{d^2 u}{dx^2} - \frac{u}{2}\frac{dp_i}{dx}\frac{dv_i}{dx} + \frac{v_i}{2}\frac{du}{dx}\frac{dp_i}{dx} = q_i(uE + v_i F)$$

(2.26) $$2pu + \frac{u^2}{2} + \frac{M_e v_e^2}{2} + \frac{v_i^2}{2} + FB - \frac{u}{2}\left(p_i \frac{dv_i}{dx} - \frac{p_e}{\omega_e}\frac{dv_e}{dx}\right) + \frac{1}{2}\frac{du}{dx}\left(v_i p_i - \frac{v_e p_e}{\omega_e}\right) = \mathcal{E}$$

Maxwell

(2.27) $$\frac{dB}{dx} = q_i(v_e - v_i).$$

If we multiply equation (2.21) by u and equation (2.22) by v_i and subtract them from equation (2.25) we are left with

$$u \frac{dp_i}{dx} + 2p_i \frac{du}{dx} = 0 \quad \text{or} \quad p_i \sim \frac{1}{u^2} .$$

Thus the ions behave adiabatically and the same is obviously true of the electrons. In spite of these adiabatic relations, the off-diagonal stress terms will still be present in our equations because of the difference in mass between the ions and electrons. If the plasma had consisted of particles of equal mass, however, our equations would have reduced to those describing the isotropic case.[6]

3. Range of Parameters.

We shall show that a necessary condition for the existence of a shock is that the parameters (integration constants) be kept within a certain range of values. These results will then be used in determining the nature of the singularities ahead of and behind the shock.

Since the solution we are seeking must attain constant values at ± ∞, the derivatives of the dependent variables must approach zero far from the shock front. Our equations will then reduce to

$$E = v_i = v_e = 0 \tag{3.1}$$

$$F = uB \tag{3.2}$$

$$u + p + \frac{B^2}{2} = 1 \tag{3.3}$$

$$2pu + \frac{u^2}{2} + FB = \mathcal{E} \tag{3.4}$$

Eliminating p between (3.3) and (3.4) and using (3.2) leads to

(3.5) $$u = \frac{1 \pm \sqrt{1 - \frac{3}{2}\mathcal{E}}}{3/2}$$

hence, it is necessary that $\mathcal{E} \leq \frac{2}{3}$. Letting $\epsilon = \sqrt{1 - \frac{3}{2}\mathcal{E}}$ we obtain

(3.6) $$u = \frac{2}{3}(1 \pm \epsilon).$$

Obviously the "+" sign will give the value of u at $-\infty$, the "-" sign at $+\infty$.

The lower limit of $\mathcal{E}$ is found by allowing B and p to approach zero ahead of the shock. It is permissible to let $B \longrightarrow 0$ since we are dealing with fast shocks and hence a "complete switch-on" phenomenon may occur.[10] With $B = p = 0$, we find $\mathcal{E} = \frac{1}{2}$ with $u = 1$ ahead of the shock and $u = \frac{1}{3}$ behind. The ranges of $\mathcal{E}$ and ϵ are, therefore,

$$\frac{1}{2} \leq \mathcal{E} \leq \frac{2}{3}$$

$$\frac{1}{2} \geq \epsilon \geq 0.$$

We note that if $\mathcal{E} = \frac{2}{3}$, then $u = \frac{2}{3}$ everywhere, i.e., we have a zero strength shock. In terms of our original variables this leads to $u^2 = 2(\frac{p}{\rho} + \frac{B^2}{2\mu\rho})$ which states, as expected, that a zero strength shock results when the velocity ahead of the shock is just equal to the fast disturbance speed ahead.

From $B = F/u$ we have

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[10] Cf. report by Friedrichs and Kranzer, pp. 30-34, cited in reference (1).

$$B = \frac{3}{2}\frac{F}{1 \pm \epsilon} \tag{3.7}$$

where the "+" is to be used ahead of the shock, the "-" sign, behind. Solving for p yields:

$$p = 1 - \frac{2}{3}(1 \pm \epsilon) - \frac{9}{8}\frac{F^2}{(1 + \epsilon^2 \pm 2\epsilon)} . \tag{3.8}$$

Since $p \geq 0$, we obtain

$$0 \leq F^2 \leq \frac{8}{27} .$$

Both ϵ and F may vary independently of each other. Furthermore, each has a different meaning since ϵ determines shock strength whereas F is essentially a measure of magnetic field strength. For example, even with $F = 0$ (no B field), there can still be a jump in u and p. However, for every value of ϵ there is an upper limit for F which is determined by the condition that $p \geq 0$. For example, when $\epsilon = 1/2$ it is necessary that $F = 0$ in order to have $p > 0$.

4. Direction Field.

We shall show that the system of equations (2.21) to (2.27) admits of a 4-dimensional direction field. More specifically, we shall obtain a system of four equations in which the first derivatives of v_e, v_i, u and B are each, separately, expressed in terms of an integrated function of these quantities.

We shall now outline the procedure for obtaining the direction field:

1) The expression $\frac{p_i}{2}\frac{dv_i}{dx} - \frac{p_e}{2\omega_e}\frac{dv_e}{dx}$ may be eliminated between (2.23) and (2.26) leading to an integrated relation of the following form:

$$\frac{u}{2}\left(p_i - \frac{p_e}{\omega_e}\right)B^2 - F\left(p_i - \frac{p_e}{\omega_e}\right)B - \left(p_i - \frac{p_e}{\omega_e}\right)\left[2u(p_e + p_i) + \frac{u^2}{2} + \frac{M_e v_e^2}{2}\right.$$
$$\left. + \frac{v_i^2}{2} + u(1-p_i-p_e-u) - \mathcal{E}\,\right] - (v_i + M_e v_e)\left(v_i p_i + \frac{v_e p_e}{\omega_e}\right) = 0.$$

If we neglect terms in products of v_i and v_e since we are dealing with weak shocks and v_i and v_e are both $O(\epsilon)$ near the singularities, this equation simplifies to

$$(4.1) \qquad p_i = \frac{1}{u}\left\{\mathcal{E} + \frac{u^2}{2} - FB - u\left(1 - \frac{B^2}{2}\right)\right\} - p_e.$$

We retain equation (2.23) which is already linear in dv_i/dx and dv_e/dx.

2) Differentiation of equation (2.24) yields a relation for d^2u/dx^2 in terms of first order derivatives. Eliminating d^2u/dx^2 between this relation and equation (2.22) leads to a linear relation between dp_e/dx, dv_e/dx and dv_i/dx. Since $p_e \sim 1/u^2$, this is, in effect, a second relation between dv_e/dx and dv_i/dx.

3) Since du/dx and dB/dx have the desired form in (2.24) and (2.27), we now have four linear equations for the four unknowns, dv_e/dx, dv_i/dx, du/dx and dB/dx. Solving, we obtain

$$\frac{dv_i}{dx} = \frac{\frac{p_e q_i}{\omega_e p_i}(F-uB) + \frac{2M_e}{P}\left(1-u-p_i-p_e-\frac{B^2}{2}\right)}{D} \tag{4.2}$$

$$\frac{dv_e}{dx} = \frac{-q_i(F-uB) + \left(\frac{2}{p_i}-\frac{2}{P}\right)\left(1-u-p_i-p_e-\frac{B^2}{2}\right)}{D} \tag{4.3}$$

$$\frac{dB}{dx} = q_i(v_e - v_i) \tag{4.4}$$

$$\frac{du}{dx} = -\frac{v_i + M_e v_e}{P/2} \tag{4.5}$$

for the direction field where $P \equiv p_i - \frac{p_e}{\omega_e}$ and $D = \frac{M_e p_i}{P} - \frac{p_e}{\omega_e}\left(\frac{1}{p_i} - \frac{1}{P}\right)$, and the ion pressure p_i is given by equation (4.1).* In writing down this direction field we have neglected terms that are of $O(\epsilon^2)$, the reason being that they will not be used at present in determining the nature of the singularities.

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* The electric field E may be calculated from equation (2.21); however, for our present purposes, we do not need to know its value.

5. Nature of the Singularities.

Linearization of the direction field in the neighborhood of the singularities yields:

$$\frac{dv_i}{dx} = \frac{1}{D_o}\,[U_{v_i}(u-u_o) + B_{v_i}(B-B_o)] \tag{5.1}$$

$$\frac{dv_e}{dx} = \frac{1}{D_o}\,[U_{v_e}(u-u_o) + B_{v_e}(B-B_o)] \tag{5.2}$$

$$\frac{dB}{dx} = q_{i_o}(v_e - v_i) \tag{5.3}$$

$$\frac{du}{dx} = -\,\frac{v_i + M_e v_e}{P_o/2} \tag{5.4}$$

where the subscript "o" may refer to either +∞ or -∞ and, furthermore,

$$U_{v_i} \equiv \frac{p_{e_o} q_{i_o} B_o}{\omega_e p_{i_o}} + \frac{2M_e}{P_o}\left(-\frac{3}{2} + \frac{\mathcal{E}}{u_o^2} - \frac{FB_o}{u_o^2}\right)$$

$$B_{v_i} \equiv \frac{p_{e_o} q_{i_o}}{\omega_e p_{i_o}}\, u_o - \frac{2M_e}{p_o}\, B_o$$

$$U_{v_e} \equiv \left(\frac{2}{p_{i_o}} - \frac{2}{P_o}\right)\left(\frac{\mathcal{E}}{u_o^2} - \frac{3}{2} - \frac{FB_o}{u_o^2}\right) + q_{i_o} B_o$$

$$B_{v_e} \equiv -\left(\frac{2}{p_{i_o}} - \frac{2}{P_o}\right) B_o + q_{i_o} u_o \quad .$$

The equation satisfied by the eigen-values is

$$\lambda^4 + \left(q_{i_o} \frac{B_{v_i}}{D_o} + \frac{U_{v_i}}{D_o P_o/2} + \frac{M_e}{P_o/2} \frac{U_{v_e}}{D_o} - \frac{q_{i_o} B_{v_e}}{D_o} \right) \lambda^2 + \frac{2q_{i_o}}{P_o D_o^2} (1 + M_e)(U_{v_e} B_{v_i} - U_{v_i} B_{v_e}) = 0. \tag{5.5}$$

Letting $\lambda^2 \equiv y$ and allowing terms multiplied by $M_e = \frac{1}{\omega_e}$ to approach zero (this is consistent with the assumption that the electrons are adiabatic), we obtain:

$$y = \frac{a}{2} \pm \frac{1}{2}\sqrt{a^2 + \frac{16\, q_{i_o}^2 M_e u_o}{P_o^2 D_o^2} \left(\frac{\mathcal{E}}{u_o^2} - \frac{3}{2}\right)} \tag{5.6}$$

or $y = a \pm b$ where $a \equiv -\left(q_{i_o} \frac{B_{v_i}}{D_o} + \frac{2U_{v_i}}{D_o P_o} + \frac{2M_e U_{v_e}}{P_o D_o} - \frac{q_{i_o} B_{v_e}}{D_o} \right)$ and $b \equiv \frac{1}{2}\sqrt{\qquad}$. Behind the shock, $\mathcal{E}/u_o^2 - 3/2 > 0$, hence b is real and also $b > a$, (note that $a > 0$). Therefore, the only stable solutions will be

$$i\lambda_1 = i\sqrt{a-b}$$

$$i\lambda_2 = -i\sqrt{a-b}$$

$$\lambda_3 = -\sqrt{a+b} \qquad (\text{with } \lambda_4 = \sqrt{a+b}\text{ , unstable})$$

where λ_1 and λ_2 correspond to a closed singularity and λ_3, to a saddle-like singularity.

Ahead of the shock, $\mathcal{E}/u_o^2 - 3/2 < 0$, but b is still real since $\mathcal{E}/u_o^2 - 3/2$ is multiplied by M_e. However, in this case we obtain $0 < b < a$ and, therefore, the stable solutions are

$$\lambda_5 = \sqrt{a+b} \text{ ,} \qquad \lambda_6 = \sqrt{a-b}$$

with

$$\lambda_7 = -\sqrt{a+b} \text{ ,} \qquad \lambda_8 = -\sqrt{a-b}$$

unstable.

Thus, the singularity ahead of the shock has the nature of a saddle-point.

In order to ascertain whether the closed singularity can open into a spiral, it would be necessary to determine the effect of the higher than linear terms.[11] We note, however, that our equations are symmetric in x, i.e., they are invariant under the transformation $x \rightarrow -x$, $E \rightarrow -E$, $v_r \rightarrow -v_r$ with B, u, p_i and p_e remaining unchanged. It seems improbable, therefore, that the closed singularity can ever become a spiral, i.e., that the solution can consist of a smooth transition from a steady state ahead of the shock to a damped oscillatory state behind. This is confirmed in the next two sections in which a method is presented whereby the solution profile can be determined analytically. As expected, it turns out to be a pulse.

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[11] See, for instance, Hurewicz, "Lectures on Ordinary Differential Equations", Technology Press and Wiley, (1958).

6. Gardner Transition Curve for Small-Amplitude Motion.*

The first step towards finding the transition curve is to introduce a new set of variables which are derived from the old by a stretching transformation; the parameter of the transformation is ϵ, the shock strength. Next, the new variables are expanded in a power series in ϵ. By keeping only terms of lowest order, it is possible to obtain a set of differential equations which are non-linear and yet simple enough to be solved. We shall show that the appropriate solution of these equations is a pulse.

The following calculations are made easier by using a slightly different set of dimensionless variables. Let B_o, n_o and u_o be the respective values of B, n and u at $x = -\infty$. Furthermore, let

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* The author is most grateful to Dr. Clifford Gardner who suggested the method and calculations described in this section.

$$\left[\frac{B_o^2}{\mu(m_e+m_i)n_o}\right]^{1/2} = u_{Alf}$$

be the Alfven speed at $x = -\infty$. The new dimensionless variables will be defined by

$$n' = \frac{n}{n_o} , \qquad B' = \frac{B}{B_o} , \qquad u' = \frac{u}{u_{Alf}}$$

$$v_i' = \frac{v_i}{u_{Alf}} , \qquad v_e' = \frac{v_e}{u_{Alf}}$$

$$p_i' = \frac{p_i}{(m_i+m_e)n_o u_{Alf}^2}$$

$$p_e' = \frac{p_e}{(m_i+m_e)n_o u_{Alf}^2}$$

$$x' = \frac{eB_o}{(m_i+m_e)u_{Alf}} x.$$

Letting $\alpha \equiv m_e/m_i + m_e$ and omitting the primes, we have

$$(6.1) \qquad \frac{d}{dx}(nu) = 0$$

$$(6.2) \qquad \frac{d}{dx}\left[u_o u + p_i + p_e - (1-\alpha)\frac{p_i}{2B}\frac{dv_i}{dx} + \alpha\frac{p_e}{2B}\frac{dv_e}{dx} + \frac{1}{2}B^2\right] = 0$$

$$(6.3) \qquad \frac{d}{dx}\left\{u_o[(1-\alpha)v_i + \alpha v_e] + (1-\alpha)\frac{p_i}{2B}\frac{du}{dx} - \alpha\frac{p_e}{2B}\frac{du}{dx}\right\} = 0$$

$$(6.4) \qquad u_o(1-\alpha)\frac{dv_i}{dx} + (1-\alpha)\frac{d}{dx}\left(\frac{p_i}{2B}\frac{du}{dx}\right) = n(u_o - Bu)$$

$$(6.5) \qquad \frac{dB}{dx} = n(v_e - v_i)$$

$$(6.6) \qquad u\frac{dp_i}{dx} + 2p_i\frac{du}{dx} = 0$$

$$(6.7) \qquad u\frac{dp_e}{dx} + 2p_e\frac{du}{dx} = 0$$

In this system, the equation corresponding to (2.21) has been omitted since it merely serves to determine E. The energy equations (6.6) and (6.7) have been written down separately for each component (instead of using the total momentum equation together with the ion energy equation as was done previously). The simplicity of their form results from the use of equations (6.2) and (6.3).

We now define new variables $\bar{u}$, $\bar{B}$, $\bar{p}_i$, etc., by means of a stretching transformation appropriate for small-amplitude motion. The correct transformation is partially indicated by relation (5.6) which shows that the wave-numbers λ_1 and λ_2 scale as $\epsilon^{1/2}$ for small ϵ. Thus the new space variable ξ will be defined as $\xi = \epsilon^{1/2}x$. Now, let the initial conditions ahead of the shock be

$$p_i = \frac{p_o}{2} = \frac{c_o^2 - 1}{4}$$

$$p_e = \frac{p_o}{2} = \frac{c_o^2 - 1}{4}$$

$$u_o = c_o(1+\epsilon)$$

where $c_o = \sqrt{2(p_o + 1/2)}$ is the fast disturbance speed. We shall assume, furthermore, that

$$u = c_o(1+\epsilon)(1+\epsilon\bar{u})$$

$$p_i = \left(\frac{c_o^2 - 1}{4}\right)(1 + \epsilon\bar{p}_i)$$

$$p_e = \left(\frac{c_o^2 - 1}{4}\right)(1 + \epsilon\bar{p}_e)$$

$$B = 1 + \epsilon\bar{B}$$

$$n = 1 + \epsilon\bar{n}$$

and, therefore,

$$v_i = c_o(1 + \epsilon)\epsilon^{3/2}\,\bar{v}_i$$

$$v_e = c_o(1 + \epsilon)\epsilon^{3/2}\,\bar{v}_e.$$

We note that at $x = -\infty$ we have $\bar{u} = \bar{v}_i = \bar{v}_e = \bar{p}_i = \bar{p}_e = \bar{B} = \bar{n} = 0$.

Substituting these expansions into the system (6.1) to (6.7) and retaining terms of the two lowest orders above the constant term, we obtain from (6.6) and (6.7):

$$(1 + \epsilon\bar{u})\frac{d\bar{p}_i}{d\xi} + 2(1 + \epsilon\bar{p}_i)\frac{d\bar{u}}{d\xi} = 0 \tag{6.8}$$

$$(1 + \epsilon\bar{u})\frac{d\bar{p}_e}{d\xi} + 2(1 + \epsilon\bar{p}_e)\frac{d\bar{u}}{d\xi} = 0. \tag{6.9}$$

This yields

$$(1+\epsilon\bar{p}_i) = (1+\epsilon\bar{p}_e) = \frac{1}{(1+\epsilon\bar{u})^2} = 1 - 2\epsilon\bar{u} + 3\epsilon^2\,\bar{u}^2$$

correct to $O(\epsilon^2)$ or

$$(6.10)\qquad \bar{p}_i = \bar{p}_e = -2\bar{u} + 3\epsilon\bar{u}^2 \quad .$$

Using (6.10) with the initial conditions, equations (6.1) to (6.5) become:

$$(6.1a)\qquad \bar{n} + \bar{u} + \epsilon\,\bar{n}\,\bar{u} = 0$$

$$(6.2a)\qquad \bar{u} + \bar{B} + \epsilon\Big\{2c_o^2\,\bar{u} + \tfrac{3}{2}\,(c_o^2 - 1)\bar{u}^2 + \tfrac{1}{2}\,\bar{B}^2$$

$$-(1-\alpha)c_o\Big(\frac{c_o^2-1}{8}\Big)\frac{d\bar{v}_i}{d\xi} + \alpha c_o\Big(\frac{c_o^2-1}{8}\Big)\frac{d\bar{v}_e}{d\xi}\Big\} = O(\epsilon^2)$$

$$(6.3a)\qquad (1-\alpha)\bar{v}_i + \alpha\bar{v}_e + \frac{(1-2\alpha)}{c_o}\,\Big(\frac{c_o^2-1}{8}\Big)\frac{d\bar{u}}{d\xi} + \epsilon\Big\{2[(1-\alpha)\bar{v}_i + \alpha\bar{v}_e]$$

$$+\frac{(1-2\alpha)}{c_o}\,\Big(\frac{c_o^2-1}{8}\Big)(1-2\bar{u}-\bar{B})\,\frac{d\bar{u}}{d\xi}\Big\} = O(\epsilon^2)$$

$$(6.4a)\qquad \bar{u} + \bar{B} + \epsilon\Big\{(1+\bar{n})(\bar{u}+\bar{B}) + \overline{uB} + c_o(1-\alpha)\,\frac{d\bar{v}_i}{d\xi}$$

$$+ (1-\alpha)\Big(\frac{c_o^2-1}{8}\Big)\,\frac{d^2u}{d\xi^2}\Big\} = O(\epsilon^2)$$

$$(6.5a)\qquad \frac{d\bar{B}}{d\xi} + c_o(\bar{v}_i - \bar{v}_e) + \epsilon c_o(\bar{n}+1)(\bar{v}_i - \bar{v}_e) = 0$$

We now assume that

$$\bar{n} = \bar{n}_1 + \epsilon\bar{n}_2 + \epsilon^2\bar{n}_3 + \dots$$

$$\bar{u} = \bar{u}_1 + \epsilon\bar{u}_2 + \dots$$

$$\bar{v}_i = \bar{v}_{i_1} + \epsilon\bar{v}_{i_2} + \dots$$

$$\bar{v}_e = \bar{v}_{e_1} + \epsilon\bar{v}_{e_2} + \dots$$

$$\bar{B} = \bar{B}_1 + \epsilon\bar{B}_2 + \dots$$

To lowest order we obtain

$$\bar{n}_1 = -\bar{u}_1 = -f(\xi)$$

$$\bar{B}_1 = -\bar{u}_1 = -f(\xi)$$

$$c_o(\bar{v}_{i_1} - \bar{v}_{e_1}) = \frac{d\bar{u}_1}{d\xi} = f'(\xi)$$

$$(1-\alpha)\bar{v}_{i_1} + \alpha\bar{v}_{e_1} = -\frac{(1-2\alpha)}{c_o}\left(\frac{c_o^2 - 1}{8}\right)\frac{d\bar{u}_1}{d\xi} =$$

$$= -\left(\frac{1-2\alpha}{c_o}\right)\left(\frac{c_o^2 - 1}{8}\right) f'(\xi).$$

To get an equation for $f(\xi)$, we substract (6.2a) from (6.4a) and insert the values of the various quantities in terms of f. Using

$$\bar{v}_{i_1} = \left[-\left(\frac{1-2\alpha}{c_o}\right)\left(\frac{c_o^2 - 1}{8}\right) + \frac{\alpha}{c_o}\right] f'$$

and

$$v_{e_1} = [-(\frac{1-2\alpha}{c_o})(\frac{c_o^2 - 1}{8}) - (\frac{1-\alpha}{c_o})]f'$$

we finally obtain:

$$\frac{d^2f}{d\xi^2} + 2Af + \frac{3}{2}Af^2 \tag{6.11}$$

where $A \equiv \dfrac{-c_o^2}{\alpha(1-\alpha) + 2\alpha(1-\alpha)(\frac{c_o^2 - 1}{8}) - (1-2\alpha)^2\left(\frac{c_o^2 - 1}{8}\right)^2}$.

The solution of equation (6.11) is easily seen to be

$$f = -2 \operatorname{sech}^2 \lambda\xi \qquad (\lambda \equiv \frac{4\sqrt{2}\, c_o}{(c_o^2 - 1)})$$

which represents a pulse.

7. Second Order Approximation to the Gardner Transition Curve.

The calculations described in this section will show that the second order terms in the system (6.1) to (6.7) do not introduce any essential modifications of the solution found in the preceeding section. More specifically, these terms do not introduce a structure into the pulse which remains simple.*

With second order terms included and using $\bar{p}_i = \bar{p}_e = -2u + 3\epsilon\bar{u}^2$ and $\alpha = 0$, equations (6.2) to (6.5) become

$$\bar{u} + \bar{B} + \epsilon\left\{\frac{3}{2}(c_o^2 - 1)\bar{u}^2 + 2c_o^2\bar{u} + \frac{\bar{B}^2}{2} - c_o\left(\frac{c_o^2-1}{8}\right)\frac{d\bar{v}_i}{d\xi}\right\} + \epsilon^2\left\{c_o^2\bar{u} - c_o\left(\frac{c_o^2-1}{8}\right)(-2\bar{u} - \bar{B} + 1)\frac{d\bar{v}_i}{d\xi}\right\} = 0 \tag{7.1}$$

$$\bar{v}_i + \frac{1}{c_o}\left(\frac{c_o^2-1}{8}\right)\frac{d\bar{u}}{d\xi} + \epsilon\left\{2\bar{v}_i + \frac{1}{c_o}\left(\frac{c_o^2-1}{8}\right)(1-\bar{B})\frac{d\bar{u}}{d\xi} - \frac{2\bar{u}}{c_o}\left(\frac{c_o^2-1}{8}\right)\frac{d\bar{u}}{d\xi}\right\} + \epsilon^2\left\{\bar{v}_i + \frac{3\bar{u}^2}{c_o}\left(\frac{c_o^2-1}{8}\right)\frac{d\bar{u}}{d\xi} + \left[\frac{1}{c_o}\left(\frac{c_o^2-1}{8}\right)\bar{B}(\bar{B}-1) - \frac{2\bar{u}}{c_o}\left(\frac{c_o^2-1}{8}\right)(1-\bar{B})\right]\frac{d\bar{u}}{d\xi}\right\} = 0 \tag{7.2}$$

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* It had been previously reported that these terms did introduce a structure (cf. Bulletin of the American Physical Society, Series II, Vol. 5, No. 4).

$$\begin{aligned} &\bar{u} + \bar{B} + \epsilon \left\{ (1+\bar{n})(\bar{u}+\bar{B}) + \overline{uB} + c_o \frac{d\bar{v}_i}{d\xi} + \left(\frac{c_o^2-1}{8}\right) \frac{d^2\bar{u}}{d\xi^2} \right\} \\ &+ \epsilon^2 \left\{ 2c_o \frac{d\bar{v}_i}{d\xi} + \left(\frac{c_o^2-1}{8}\right) \frac{d}{d\xi} \left[(-2\bar{u}-\bar{B}+1) \frac{d\bar{u}}{d\xi} \right] + \bar{n}(\bar{u}+\bar{B}) \right. \\ &\left. + (\bar{n}+1)\overline{uB} \right\} = 0 \end{aligned} \tag{7.3}$$

$$\frac{d\bar{B}}{d\xi} - c_o(\bar{v}_e - \bar{v}_i) - c_o\epsilon(\bar{n}+1)(\bar{v}_e - \bar{v}_i) - \epsilon^2 \bar{n} c_o (\bar{v}_e - \bar{v}_i) = 0. \tag{7.4}$$

Equations (7.2) and (7.3) yield, respectively:

$$\bar{v}_i = \frac{1}{c_o} \left(\frac{c_o^2-1}{8}\right) \left\{ 1+\epsilon[1+\bar{B}+2\bar{u}] \right\} \frac{d\bar{u}}{d\xi}$$

and

$$\bar{B} = -\bar{u} + \epsilon\bar{u}^2 - \epsilon^2\bar{u}^3.$$

Substituting these expressions into equations (7.1) leads to the following second order differential equation:

$$\begin{aligned} &\left(\frac{c_o^2-1}{8}\right)^2 [1-2\epsilon\bar{u}] \frac{d^2\bar{u}}{d\xi^2} - \epsilon \left(\frac{c_o^2-1}{8}\right)^2 \left(\frac{d\bar{u}}{d\xi}\right)^2 + 2c_o^2\,\bar{u} \\ &+ \frac{3}{2} c_o^2\,\bar{u}^2 + \epsilon(c_o^2\,\bar{u} - 2\bar{u}^3) = 0 \end{aligned} \tag{7.5}$$

Setting $\bar{u} = u^{(1)} + \epsilon u^{(2)} + \epsilon^2 u^{(3)} + \ldots,$ we find

$$(7.6)\qquad \frac{d^2u^{(2)}}{d\xi^2} + \left(\frac{8c_o}{c_o^2-1}\right)^2 (2+3u^{(1)})u^{(2)} = 2u^{(1)^3} - c_o^2\, u^{(1)}$$
$$+ \left(\frac{c_o^2-1}{8}\right)^2 \left(\frac{du^{(1)}}{d\xi}\right)^2 + 2\left(\frac{c_o^2-1}{8}\right)^2 u^{(1)} \frac{d^2u^{(1)}}{d\xi^2}$$

where, obviously, $u^{(1)} = -2\,\mathrm{sech}^2\lambda\xi \quad (\lambda = \frac{4\sqrt{2}\,c_o}{(c_o^2-1)})$.

We shall first find the solutions of the homogeneous equation. Using the standard technique for reducing a second order linear equation to one of first order, we set $u^{(2)'}/u^{(2)} = W(\xi)$ and obtain the Riccati equation

$$(7.7)\qquad \frac{dW}{d\xi} + W^2 + K^2(2+3u^{(1)}) = 0$$

where

$$K^2 \equiv \left(\frac{8c_o}{c_o^2-1}\right)^2 .$$

We shall be interested in solutions valid near $\xi = 0$ and since we want $du^{(2)}/d\xi = 0$ at $\xi = 0$, W^2 will certainly be small in this region. Thus, neglecting W^2, we obtain an one approximate solution of the homogeneous equation:

$$u^{(2)} = e^{-K^2\xi^2}(\cosh \lambda\xi)^{6K^2/\lambda^2} \equiv v.$$

The second solution w may be found by setting $w = v \int Z d\xi$ where $Z(\xi)$ is an unknown function whose form is determined by substituting w into the original equation. We find

$w = v \int dz/v^2$.

The particular solution that we are seeking has the form $U = c_1(\xi)v(\xi) + c_2(\xi)w(\xi)$ where

$$c_1(\xi) = - \int f(\xi)w(\xi)d\xi$$

and

$$c_2(\xi) = \int f(\xi)v(\xi)d\xi$$

with

$$f(\xi) = 2u^{(1)^3} - c_o^2\, u^{(1)} + \left(\frac{c_o^2-1}{8}\right)^2 \left(\frac{du^{(1)}}{d\xi}\right)^2 + 2\left(\frac{c_o^2-1}{8}\right)^2 u^{(1)} \frac{d^2u^{(1)}}{d\xi^2} .$$

At $\xi = 0$ it is easy to see that $U(\xi)$ has a maximum since $dU/d\xi = 0$ and the sign of $d^2U/d\xi^2$ is determined solely by the sign of $f(0)$ which is negative. More explicitly, we find that

$$\frac{dU}{d\xi} = -\left\{-2K^2\xi(\cosh \lambda\xi)^{\frac{6K^2}{\lambda^2}} + \frac{6K^2}{\lambda}(\cosh \lambda\xi)^{\frac{6K^2}{\lambda^2}-1} \sinh \lambda\xi\right\} e^{-K^2\xi^2} \cdot \int_0^\xi f(z)v(z)dz \int_0^z \frac{dy}{v^2(y)} + \left\{\frac{1}{v(\xi)} + e^{-K^2\xi^2}\left[-2K^2\xi(\cosh \lambda\xi)^{\frac{6K^2}{\lambda^2}} + \frac{6K^2}{\lambda}(\cosh \lambda\xi)^{\frac{6K^2}{\lambda^2}-1} \sinh \lambda\xi\right]\right\}\left[\int_0^\xi \frac{d\xi}{v^2(\xi)}\right]\left[\int_0^\xi f(\xi)e^{-K^2\xi^2}(\cosh \lambda\xi)^{\frac{6K^2}{\lambda^2}} d\xi\right] \tag{7.8}$$

If $K^2 << 1$ and $\lambda << 1$, both of which are satisfied if $c_o^2 - 1 >> c_o$ or $2p_o >> c_o$, then the various functions appearing in (7.8) will be well approximated by the first few terms in their power series expansions. Retaining terms up to and including ξ^3, we obtain finally*

$$\frac{dU}{d\xi} = \xi(2K^2\xi^2 + 1)$$

which can only be zero at $\xi = 0$. This is the case that is of interest in thermonuclear research.

On the other hand, if $K^2 >> 1$ and $\lambda >> 1$, the first few terms of the expansion are no longer sufficient to determine the zeros of $dU/d\xi$. However, $K^2 >> 1$ means $2p_o << c_o$ which corresponds to the cold plasma case and here we know that the solution is a simple pulse.

It is not yet clear whether these results may be extrapolated to the intermediate case of a cool (not cold) plasma.

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* The W^2 term in the Riccati equation contributes terms in ξ^5 to $dU/d\xi$; therefore, to this approximation, we may neglect it.

Appendix. Evaluation of the Stress Terms.

The equation of transfer, obtained from the Boltzmann equation by taking moments over velocity space, form an undetermined system. This situation may be corrected by assuming that certain relationships exist between the higher order and the lower order moments and their derivatives. The form of these relationships will obviously depend on the properties of the medium. In a collision-dominated gas, for example, where the departure of the distribution function from the Maxwellian is small, the Navier-Stokes relations are appropriate. In a collisionless plasma, on the other hand, we shall assume, in accordance with guiding-center theory, that the electric and magnetic fields are dominant.

In the ensuing calculations, the conservation equations will be used in the form derived by Kolodner.[12] We shall only need the following two equations:

$$(\mathrm{I}.1) \qquad De_r + \frac{\partial}{\partial x} \cdot Q_r + P_r : \frac{\partial u}{\partial x} + \frac{\rho_r}{\rho} F_r \cdot U_r = 0$$

$$(\mathrm{I}.2) \qquad D\left\{P_r\right\} + \left\{\frac{\partial}{\partial x} F_r^{(3)}\right\} + 2\left\{P_r \cdot \frac{\partial u}{\partial x}\right\} - 2\gamma_r\left\{P_r \times B\right\} + 2\,\frac{\rho_r}{\rho}\left\{F_r\, U_r\right\} = 0$$

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[12] Kolodner, Ignace I., "Moment Description of Gas Mixtures - I", NYO-7980, pg. 19 (1957).

where

$$D = \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \cdot u$$

$\mathcal{U}_r = u_r - u$ (here u is a vector; in terms of our previous notation, $u^i = u$, $u^j = v$)

$$F_r = [- \frac{\partial}{\partial x} \cdot P + (q - \gamma_r)E + (J - \gamma_r \; \rho u) \times B]$$

$$\left\{a^{ij}\right\} = \frac{1}{2}(a^{ij} + a^{ji}) - \frac{1}{3} a^{KK} \delta_{ij}$$

$$F_r^{(3)} = m_r n_r \overline{c^{i_1} c^{i_2} c^{i_3}}$$

therefore

$$P_r = F_r^{ij} \equiv P_r^{ij} .$$

Now assume that

$$E \sim O(\epsilon^{-1}) , \qquad B \sim O(\epsilon^{-1})$$

$$P_r \sim O(\epsilon^{0}) , \qquad u_r \sim O(\epsilon^{0})$$

$$\frac{\partial u_r}{\partial x} \sim O(\epsilon^{0}), \qquad \frac{\partial p_r}{\partial x} \sim O(\epsilon^{0})$$

$$\Pi_r^{jj} \sim O(\epsilon), \qquad \Pi_r^{ii} \sim O(\epsilon), \qquad F_r^{(3)} \sim O(\epsilon)$$

where $\Pi_r^{ii} = P_r^{ii} - p_r$ and $\Pi_r^{jj} = P_r^{jj} - p_r$, and ϵ is a parameter of smallness. Equating terms of $O(\epsilon^{-1})$, we obtain, for out geometry,

$$P_r^{ii} = P_r^{jj} = p_r$$

and

$$P_r^{ij} = P_r^{ji} = 0.$$

Next, equating terms of $O(\epsilon^{o})$ leads to

$$D\left\{P_r\right\} + 2\left\{P_r \cdot \frac{\partial u}{\partial x}\right\} - 2\gamma_r\left\{P_r \times B\right\} = 0.$$

It is easily shown that the first term of this equation vanishes and that the remainder yields

$$P_r^{ij} = \frac{1}{2\omega_r} p_r \frac{du_r}{dx} = P_r^{ji}$$

$$P_r^{ii} = p_r - \frac{p_r}{2\omega_r} \frac{dv_r}{dx}$$

$$P_r^{jj} = p_r + \frac{p_r}{2\omega_r} \frac{dv_r}{dx} \quad .$$

Since the relations for Q_r may be obtained in an entirely similar fashion from the transfer equations for the next order moments, we shall not repeat the argument here.

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